

ADDENDUM TO THE PAPER ON PARTIALLY STABLE ALGEBRAS

BY
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I regret to announce that there is a serious error in my paper in these Transactions, volume 84, pp. 430–443. The error was discovered by Louis Kokoris who found that on line 8 of page 434 the expression given as $4[g(bz)](az)$ should have been $4[g(az)](bz)$. As a consequence the computation of $P(z, g, az, b)$ yields nothing, the proof of formula (30) is not valid, and the important Lemma 9 is not proved. Thus the paper does not give a proof of its major result stated as Theorem 1.

Nevertheless, the theorems of the paper are all correct and we shall provide a revision of the proof here. This revised proof has been checked by Louis Kokoris to whom the author wishes to express his great thanks.

We observe first that the equation

$$(29) \quad gS_{ab} = gS_aS_b - (wa)[z\phi_g(b)]$$

occurs before the error and is correct. The results of §4 on pages 434 and 435 depend on Lemma 9, and their use must be postponed until we apply them in a new Lemma 25. Equation (41) is correct as proved and will become a part of a corrected version of Lemma 17. The following result will be used here and is proved correctly in the paper.

LEMMA 14. *Let a and b be in $\mathfrak{B}$ and g be in $\mathfrak{G}$. Then*

$$(42) \quad g[(wa)(bz)] = b\phi_g(a) - \phi_gS_b(a).$$

Lemma 17 uses Theorem 1 and so needs revision. The computation of $P(wa, bz, c, w)$ of its proof actually yields $4a(bc) - 3b(ca) - c(ab) = \phi_h(a) - 4\phi_g(c)$. The interchange of a and c implies the interchange of g and h and results in the formula $4c(ba) - 3b(ca) - a(cb) = \phi_g(c) - 4\phi_h(a)$. The quantity $\phi_h(a)$ can now be eliminated and we obtain the second equation in a relation which we shall number (41R). The proof of the first relation is exactly as in the paper.

LEMMA 17R. *Let $g = (wa)(bz)$, where a and b are in $\mathfrak{B}$. Then*

$$(41R) \quad f_g(c) = 0, \quad \phi_g(c) = (ac)b - a(cb)$$

for every c of $\mathfrak{B}$.

Since Theorem 1 is still not valid at this point we shall require the following result.

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LEMMA A. *The algebra $\mathfrak{B}$ is a special Jordan algebra such that $\mathfrak{B} = e\mathfrak{F} + \mathfrak{N}$, where e is the unity element of the algebra $\mathfrak{A}$, and $\mathfrak{N}$ is the radical of $\mathfrak{B}$.*

For Lemma 2 states that the algebra $\mathfrak{B}$ is isomorphic to the algebra $\mathfrak{A}_u(1) = \mathfrak{B}u$. It is also known⁽¹⁾ that the mapping $a_1 \rightarrow S_{1/2}(a_1)$, defined by the multiplication $x_{1/2}a_1 = x_{1/2}S_{1/2}(a_1)$ in our u -stable case, is a homomorphic mapping of $\mathfrak{B}u$ onto the special Jordan algebra of all of the linear transformations $S_{1/2}(a_1)$ where a_1 ranges over all elements of $\mathfrak{B}u$. These elements all have the form $a_1 = au$ for a in $\mathfrak{B}$, and if $S_{1/2}(a_1) = 0$ then $wS_{1/2}(a_1) = w(au) = 0$. However, Lemma 3 states that $w(au) = 2^{-1}(wa)$. Hence the kernel of our homomorphism consists of all elements au such that $wa = 0$, $w(wa) = a = 0$. It follows immediately that $\mathfrak{B}$ is a special Jordan algebra. The basic assumption that $\mathfrak{A}$ has degree two implies that the only idempotent of $\mathfrak{B}u$ is u , the only idempotent of $\mathfrak{B}$ is its unity element e . It is then known⁽²⁾ that $\mathfrak{B} = e\mathfrak{F} + \mathfrak{N}$ as desired.

We are now able to use Lemma 17R, Lemma A, and Lemma 4 to derive the following result.

LEMMA B. *The inclusion relation $[(w\mathfrak{B})(\mathfrak{B}z)](\mathfrak{B}z) \subseteq w\mathfrak{N}$ holds.*

For, we can always write every element a of $\mathfrak{B}$ in the form $a = \alpha e + c$ where α is in $\mathfrak{F}$ and c is in $\mathfrak{N}$. Since $w(bz) = (wa)z = 0$ for every a and b of $\mathfrak{B}$ we also write b in the form $b = \beta e + d$ with d in $\mathfrak{N}$ and have $(wa)(bz) = (wc)(dz)$, that is,

$$(1A) \quad (w\mathfrak{B})(\mathfrak{B}z) = (w\mathfrak{N})(\mathfrak{N}z).$$

But if $g = (wa)(bz)$ we use (41R) to see that $[(wa)(bz)](tz) = -w\phi_g(t)$ where $\phi_g(t) = (at)b - a(tb)$. By (1A) we can take both a and b to be in $\mathfrak{N}$, $\phi_g(t)$ is in $\mathfrak{N}$ and our proof is complete.

Lemmas 18 and 19 of our original version state that $[w(bc)](az) = b[(wc)(az)] + c[(wb)(az)]$ and that consequently

$$(47) \quad f_g(a)[(wa)(bz)] = 0$$

for every a and b of $\mathfrak{B}$. However, we cannot use Lemma 16 to conclude that $(w\mathfrak{B})(\mathfrak{B}z) = 0$ and we shall, in fact, be forced to delete Lemma 16. The proof of the relation

$$(45) \quad [wf_g(a)](zb) + [wf_g(b)](za) = 0$$

is correct and this relation is used later. We now use our results to obtain a corrected proof of the following lemma.

LEMMA 20. *Let there exist a nonsingular element $c = f_g(a)$ for some g in $\mathfrak{G}$ and a in $\mathfrak{B}$. Then $(w\mathfrak{B})(\mathfrak{B}z) = 0$.*

⁽¹⁾ See Lemma 1 of *A theory of power-associative commutative algebras*, these Transactions, vol. 69 (1950) pp. 503-527.

⁽²⁾ Ibid., Theorem 2.

For (45) is equivalent to $(wc)(az) = 0$ if $c = f_\sigma(a)$. Lemma 18 then implies that in this case $[w(bc)](az) = c[(wb)(az)]$. However, (10) states that $(wb)(az) = -(wa)(bz)$ and so (47) implies that $[w(bc)](az) = 0$. If c is any nonsingular element of the special Jordan algebra $\mathfrak{B}$ we can write $c = \gamma e + d$ where d is in $\mathfrak{N}$ and γ is in $\mathfrak{F}$. It is known⁽³⁾ that the corresponding right multiplication R_d of the algebra $\mathfrak{B}$ is nilpotent and it follows that the right multiplication $R_c = \gamma I + R_d$ is nonsingular. Hence the equation $bc = t$ has a solution b in $\mathfrak{B}$ for every t of $\mathfrak{B}$. It follows that our assumption that $c = f_\sigma(a)$ is nonsingular implies that $(wt)(az) = 0$ for every t of $\mathfrak{B}$ and every a of $\mathfrak{B}$ such that $f_\sigma(a)$ is nonsingular. It remains to consider the singular elements $f_\sigma(m)$. The sum $f_\sigma(m) + f_\sigma(a) = f_\sigma(a+m)$ is then nonsingular, that is, is not in the radical $\mathfrak{N}$. Then our proof implies that $(wt)[(a+m)z] = 0 = (wt)(az) + (wt)(mz) = (wt)(mz)$ as desired, and we have completed a proof of the relation $(w\mathfrak{B})(\mathfrak{B}z) = 0$ in the case of a nonsingular value of the function $f_\sigma(a)$.

We are now able to combine our results so as to restore the validity of the first of our basic theorems.

LEMMA C. *Let a simple commutative power-associative algebra $\mathfrak{A}$ of degree two contain an element g of $\mathfrak{G}$ and an element a of $\mathfrak{B}$ such that $f_\sigma(a)$ is nonsingular. Then Lemmas 9, 10, 11, 12, 13, and 15 are all correct and Theorem 1 is correct.*

For Lemma 20 implies that the relation (29) reduces to $S_{ab} = S_a S_b = S_b S_a$ and this and Lemma 20 give Lemma 9 completely. Thus all of the results of §4 are now correct.

Lemma C reduces our study to what we have called the "singular case," and to the possible case where every $f_\sigma(a)$ is singular but some $\phi_\sigma(a)$ is nonsingular. We shall actually be able to show that the latter case cannot occur and indeed that every $\phi_\sigma(a)$ is in $\mathfrak{N}$. Our first new result in this direction follows.

LEMMA D. *Let a , b , and c be in $\mathfrak{B}$ so that $g = (wa)(bz)$ and $h = (wa)(cz)$ are in G . Then gh is in $\mathfrak{N}$.*

By (1A) we can assume that a , b , and c are all in $\mathfrak{N}$. Compute $P(wa, wa, bz, cz)$ to see that $8gh + 4a^2(bc) = 2(wa)[g(cz) + h(bz) + (wa)(bc)] + (bz)[a^2(cz) + 2h(wa)] + (cz)[a^2(bz) + 2g(wa)]$. Then

$$\begin{aligned} 8gh = & -4a^2(bc) + 2(wa)[-w\phi_\sigma(c) - w\phi_\sigma(b) + w \cdot a(bc)] + b(a^2c) \\ & + 2b[\phi_\sigma(a) + z f_\sigma(a)] + c(a^2b) + 2c[\phi_\sigma(a) + z f_\sigma(a)]. \end{aligned}$$

Since a , b , and c are all in the radical $\mathfrak{N}$ of $\mathfrak{B}$ all terms on the right side of this equation are in $\mathfrak{N} + \mathfrak{N}z$. But gh is in $\mathfrak{B}$ and hence is in $\mathfrak{N}$.

⁽³⁾ See Theorem 1 of the author's *A structure theory for Jordan algebras*, Ann. of Math. vol. 48 (1947) pp. 546-567.

We now use the relation (42) in the form

$$(2A) \quad h[(wa)(tz)] = t\phi_h(a) - \phi_{hS_t}(a),$$

where h is any element of $\mathfrak{G}$, and a and t are arbitrary elements of $\mathfrak{B}$. Assume that

$$(3A) \quad h = (wa)(dz)$$

for the same a and some d in $\mathfrak{B}$ and use Lemma B to see that $\phi_h(a)$ is in $\mathfrak{N}$ and so $t\phi_h(a)$ is in $\mathfrak{N}$. By Lemma D we know that the left member of (2A) is in $\mathfrak{N}$. Thus (2A) implies the following result.

LEMMA E. *Let a , d , and t be any elements of $\mathfrak{N}$ and $h = (wa)(dz)$. Then $\phi_{hS_t}(a)$ is in $\mathfrak{N}$.*

There is a minor error in the statement of Lemma 21 which actually does not affect the argument but which should be corrected as follows. We compute $P(wa, z, g, g)$ and see that

$$\begin{aligned} 0 &= 2g[(wa)g \cdot z] + (wa)(g^2z) = 2g[f_\sigma(a) \cdot z + \phi_\sigma(a)] + (wa)(g^2z) \\ &= -2w\{f_\sigma[\phi_\sigma(a)] + \phi_\sigma[f_\sigma(a)]\} + 2gS_c + (wa)(g^2z) \end{aligned}$$

where $c = \phi_\sigma(a)$. This yields the corrected version of Lemma 21 which we now state.

LEMMA 21R. *Let a be in $\mathfrak{B}$, g be in $\mathfrak{G}$, and $c = \phi_\sigma(a)$. Then*

$$(49R) \quad 2gS_c = -(wa)(g^2z), \quad f_\sigma[\phi_\sigma(a)] + \phi_\sigma[f_\sigma(a)] = 0.$$

We have now obtained the key elements in a proof of the singularity of the function $\phi_\sigma(a)$.

LEMMA F. *The elements $\phi_\sigma(a)$ are in $\mathfrak{N}$ for every g of $\mathfrak{G}$ and a of $\mathfrak{B}$.*

For assume that there does exist an element g of $\mathfrak{G}$ and an element a of $\mathfrak{B}$ such that $c = \phi_\sigma(a)$ is nonsingular. By (49R) we have

$$(4A) \quad h = gS_c = (wa)(dz)$$

where $d = -1/2g^2$ is in $\mathfrak{B}$. We now use (29) to see that if $t = c^{-1}$ then $gS_{ct} = gS_\sigma = g$, and so

$$(5A) \quad gS_\sigma S_t = hS_t = g - (wc)[z\phi_\sigma(t)].$$

Hence

$$(6A) \quad g = hS_t + k$$

where k is in $(w\mathfrak{B})(\mathfrak{B}z)$. By (4A) and Lemma E we know that $\phi_{hS_t}(a)$ is in $\mathfrak{N}$. By Lemma B we know that $\phi_k(a)$ is in $\mathfrak{N}$. But then (6A) implies that $\phi_\sigma(a) = \phi_{hS_t}(a) + \phi_k(a) = c$ is in $\mathfrak{N}$, contrary to hypothesis.

We are now ready to turn to the discussion of the singular case of Section 6. The proof there makes some slight and unnecessary use of Lemma 9 and we correct the argument as follows. Note that Lemma 9 cannot be used since we have only proved its validity in the nonsingular case. We observe that

$$(7A) \quad \mathfrak{A} = \mathfrak{B} + \mathfrak{B}z + w\mathfrak{B} + \mathfrak{G}$$

and we now write

$$(8A) \quad \mathfrak{M} = \mathfrak{N} + \mathfrak{N}z,$$

so that $\mathfrak{M}$ is the radical of the Jordan algebra $\mathfrak{C} = \mathfrak{A}_u(1) + \mathfrak{A}_u(0) = \mathfrak{B} + \mathfrak{B}z$. Then the combination of the hypothesis that every $f_a(a)$ is in $\mathfrak{N}$ and Lemma F is equivalent to the inclusion relations

$$(8A) \quad (w\mathfrak{B})\mathfrak{G} \subseteq \mathfrak{M}, \quad \mathfrak{G}(\mathfrak{B}z) \subseteq w\mathfrak{N}.$$

Define $\mathfrak{G}^*$ to be the vector spanned by all elements of the form gS_a for g in $\mathfrak{G}$ and a in $\mathfrak{N}$. Then Lemma F implies that

$$(9A) \quad \mathfrak{G}\mathfrak{M} \subseteq \mathfrak{G}^* + w\mathfrak{N}, \quad \mathfrak{G}^*\mathfrak{B} = \mathfrak{G}^* + \mathfrak{G}^*\mathfrak{N} \subseteq \mathfrak{G}^*.$$

Also Lemma F and (42) imply that

$$(10A) \quad \mathfrak{G}[(w\mathfrak{B})(\mathfrak{B}z)] \subseteq \mathfrak{N}.$$

Then (29) implies that, if we write $x \equiv y$ for x and y in $\mathfrak{B}$ whenever $x - y$ is in $\mathfrak{N}$, we have the property

$$(11A) \quad (gS_{ab})h \equiv (gS_aS_b)h$$

for every g and h of $\mathfrak{G}$ and a and b of $\mathfrak{N}$. We now have all of the results of (55)–(60), the argument which forms the last paragraph of page 440 is valid, and the relation

$$(12A) \quad \mathfrak{G}\mathfrak{G}^* \subseteq \mathfrak{N}$$

is correct.

We are now able to prove that the space

$$(13A) \quad \mathfrak{S} = w\mathfrak{N} + \mathfrak{G}^* + (w\mathfrak{B})(\mathfrak{B}z) + \mathfrak{M}$$

is an ideal of $\mathfrak{A}$. Indeed $\mathfrak{A}\mathfrak{M} = \mathfrak{M}\mathfrak{M} + (w\mathfrak{B})\mathfrak{M} + \mathfrak{G}\mathfrak{M} \subseteq \mathfrak{M} + w\mathfrak{N} + (w\mathfrak{B})(\mathfrak{B}z) + \mathfrak{G}^* \subseteq \mathfrak{S}$. Also $\mathfrak{A}(w\mathfrak{N}) = \mathfrak{B}(w\mathfrak{N}) + (\mathfrak{B}z)(w\mathfrak{N}) + \mathfrak{G}(w\mathfrak{N}) + (w\mathfrak{B})(w\mathfrak{N}) \subseteq w\mathfrak{N} + (w\mathfrak{B})(\mathfrak{B}z) + \mathfrak{M} \subseteq \mathfrak{S}$. We see next that $\mathfrak{A}\mathfrak{G}^* = \mathfrak{B}\mathfrak{G}^* + (\mathfrak{B}z)\mathfrak{G}^* + (w\mathfrak{B})\mathfrak{G}^* + \mathfrak{G}\mathfrak{G}^* \subseteq \mathfrak{G}^* + w\mathfrak{N} + \mathfrak{M} \subseteq \mathfrak{S}$, and that $\mathfrak{A}[(w\mathfrak{B})(\mathfrak{B}z)] = (w\mathfrak{B})(\mathfrak{B}z) + \mathfrak{N}[(w\mathfrak{B})(\mathfrak{B}z)] + (\mathfrak{B}z)[(w\mathfrak{B})(\mathfrak{B}z)] + (w\mathfrak{B})[(w\mathfrak{B})(\mathfrak{B}z)] + (w\mathfrak{B})\mathfrak{G} \subseteq (w\mathfrak{B})(\mathfrak{B}z) + \mathfrak{G}^* + w\mathfrak{N} + \mathfrak{M} \subseteq \mathfrak{S}$ and our proof that $\mathfrak{S}$ is an ideal of $\mathfrak{A}$ is complete. Since $\mathfrak{S}$ does not contain the idempotent e the hypothesis that $\mathfrak{A}$ is simple implies that $\mathfrak{M} = 0$, $\mathfrak{B} = e\mathfrak{F}$, and so $\mathfrak{A}$ is actually a Jordan algebra in this case.

This completes our revision but we take this opportunity to correct the following misprints:

- p. 431, line -4. Should read $w[(ab)g - w\phi_\theta(a) \cdot (bz) - w\phi_\theta(b) \cdot (az)]$.
- p. 432, line 3. Delete $-4af_\theta(b)$ at end of sentence.
- p. 432, formula (13). Replace $\phi_\theta(a)$ by $\phi_\theta(ab)$.
- p. 432, line 3 below (13). Replace first $-$ by $+$.
- p. 432, line 4 below (13). Replace first $-$ by $+$.
- p. 433, line 11. Replace last gS_aS_b by gS_bS_a .
- p. 435, Lemma 13. Replace M by $\mathfrak{M}$.
- p. 436, line below (44). Replace $\mathfrak{A}$ by $\mathfrak{S}$.
- p. 436, line 7 below (44). Add $+(ab)(zc)$ at end of line.
- p. 436, line 8 below (44). Add $+(ab)(zc)$ before "Since."
- p. 436, line 10 below (44). Add $+(ab)(zc)$ before "is in."
- p. 440, formula (54). Replace $\subseteq \mathfrak{G}$ by $\subseteq \mathfrak{M}$.

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